



The Ability to Prove Real Analysis Problems: Investigation with APOS Conception

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Abstract

Proving is an important ability for pre-service mathematics teachers, but previous studies have rarely examined it in Real Analysis using the APOS framework, which is essential for revealing their cognitive development in constructing mathematical proofs. Real Analysis is one of the courses that develops proving ability, but their level of ability in prove problems based on the APOS conception remains unclear. Therefore, this study aims to analyze pre-service mathematics teachers' proving ability in real analysis based on the APOS conception framework. This study uses a qualitative approach with 59 university students as participants. Data were collected from a test with four proof problems designed to represent different levels of reasoning in Real Analysis. The data were then analyzed using content and thematic analysis techniques guided by APOS indicators (action, process, object, and schema) to identify students' cognitive structures in proving. The results indicate that students proved the problem without deep reflection. They did not recognize which method was required. They used some examples to prove a statement. They had not yet encapsulated the process of proof as a conceptual object. Thus, this study concludes that the ability of pre-service mathematics teachers to prove Real Analysis problems using the APOS conception approach was that most of them are still in the action conception, only a few have reached the process conception, and none have fulfilled the object and schema conception. These findings indicate that most students operate at the action level, relying on procedural manipulation without conceptual understanding, while only a few demonstrate process-level thinking by beginning to internalize proof structures. This suggests that students' difficulties stem from a lack of conceptual understanding and insufficient experience in constructing formal proofs. Therefore, the curriculum should be designed to strengthen prerequisite abilities before studying Real Analysis and to develop a learning model that enables students to prove various Real Analysis problems. This condition reflects the need for instructional strategies that facilitate the transition from action to higher APOS levels.

Keywords: APOS conception; proving ability; real analysis problems



INTRODUCTION

Proving is a fundamental ability for mathematics students and pre-service mathematics teachers, as proof is central to mathematics and a distinguishing characteristic of the discipline (Coe & Ruthven, 1994; Oflaz, Bulut, & Akcakin, 2016; Rizos & Gkrekas, 2023; Rocha, 2019). It plays a central role in verifying, constructing, and communicating mathematical ideas, while also developing students' understanding, reasoning, and argumentation. Early exposure to proofs fosters rigor and accuracy in mathematical thinking, and the process of constructing proofs cultivates resource-based cognitive skills, including mathematical, methodological, and strategic knowledge, as well as problem-solving abilities (Sommerhoff et al., 2021; Stylianides & Stylianides, 2022; Stylianides et al., 2024).

However, despite its importance, various studies report that many students still struggle to construct effective proofs. Weber (2001) found that these difficulties stemmed from a lack of conceptual understanding and strategic knowledge in constructing proofs. Furthermore, Harel & Sowder (1998) found that many students lack a well-developed proof schema, making it difficult to distinguish between valid and invalid arguments. Another study by Chamberlain & Vidakovic (2021) also reported that constructing formal proofs remains a significant cognitive challenge for students. Furthermore, a systematic review by Stylianides et al. (2024) showed that difficulties in understanding and constructing mathematical proofs persist from elementary school through college. These findings are reinforced by research by Syamsuri (2016), which shows that there are errors in compiling proofs, as well as Mukuka & Tatira (2025), which revealed that the understanding of proofs in pre-service mathematics teachers is still relatively low. These challenges highlight the need to explore specific mathematical contexts in college courses where proof construction is pivotal.

One course that develops proof-writing ability is Real Analysis. The course is mandatory material that must be studied by mathematics students and pre-service mathematics teachers because it provides opportunities to develop this ability (Umgelter & Geisler, 2023). In the Real Analysis book (Bartle & Sherbert, 2011), the exercises are dominated by proof-based problems. Real Analysis problems predominantly require formal proofs, often indicated by terms such as "Prove That" or "Show That". "Prove That" requires formal proof, whereas "Show That" allows less formal reasoning (CadwalladerOlsker, 2011). To solve Real Analysis problems effectively, reasoning is needed, because this ability requires exploration of ideas and knowledge that involves drawing conclusions by connecting facts and using deductive thinking (Isnani et al., 2020). Essentially, reasoning is the foundation on which proofs are built, because proof is the act of providing logical and rigorous arguments to establish the truth of a mathematical statement (Hanna, 2020). Mathematical connections are also needed in constructing proofs because they play an important role in the argumentation process, helping students construct and identify arguments and ensure the proof's results (Rodríguez-Nieto et al., 2023). In constructing proofs, students are encouraged to break down complex problems into smaller parts so that the problems faced are more manageable, identify patterns and regularities in mathematical problems, focus on the important elements of a problem while ignoring irrelevant details, and develop step-by-step procedures for solving problems (Nurlaelah et al., 2024).

Therefore, identifying students' difficulties in understanding how they think and construct proofs is crucial. APOS theory offers a structured framework for investigating these cognitive processes systematically. When constructing mathematical proofs, students develop cognitive structures through several stages. According to APOS theory, there are four conceptions. First, the *Action conceptions*, students perform specific mathematical actions required for the proof. Second, the *Process conceptions*, they understand the logical sequence and reasoning behind these actions. Third, the *Object conceptions*, students recognize the results of their actions as mathematical objects. Finally, in the schema conception, they integrate these actions and processes into a coherent understanding of the proof (Syamsuri & Marethi, 2018). Besides that, *in the action conception*, an individual applies the steps of the proof directly without deep reflection on why the steps work. An example is applying a memorized definition or theorem to reach a conclusion. *In the process conception*, an individual begins to understand the logical sequence of a proof and can imagine how it works. They can follow a logical argument from premise to conclusion, but does not yet consider the proof as a single entity. *In the object conception*, an individual condenses the entire proof process into a single concept or "object". This means that they

no longer simply follow the steps, but can see the entire proof as a cohesive, manipulable whole. *In the schema conception*, an individual develops a comprehensive understanding of when and how to use different types of proof, as well as how to connect different mathematical concepts. This includes an understanding of logic, proof techniques, and connections between theorems (Herlina, 2013; Septiati, 2024).

This discussion demonstrates how APOS conceptions manifest in different proof contexts. At the initial level, students tend to apply known properties or procedures directly. As they advance, they begin to justify reasoning more formally and recognize logical relationships in proofs (Syamsuri & Marethi, 2018). At higher levels, students view proofs as coherent structures that can be compared and applied to similar problems, and they can flexibly identify and apply appropriate strategies, such as proof by contradiction, without explicit guidance (Chamberlain & Vidakovic, 2021; Koyunkaya & Boz-Yaman, 2023; Mulyono, 2011; Shinariko et al., 2022; Syamsuri, 2016). These observations indicate that students' proof construction develops from performing procedural steps to applying integrated and flexible reasoning across different proof problems, illustrating variations in thinking across contexts.

APOS theory is appropriate in this study because it describes students' cognitive development in a structured way, from performing procedures to constructing integrated mathematical understanding (Arnon et al., 2014; Mulyono, 2011). This makes APOS particularly relevant for analyzing students' proof construction in abstract courses such as Real Analysis (Syamsuri & Marethi, 2018). Although previous studies have discussed students' proof abilities and the use of APOS theory, most of them either examine APOS in general mathematical contexts or review proof and proving across educational levels (Sefik et al., 2021; G. J. Stylianides et al., 2024), without specifically focusing on proof construction in Real Analysis contexts. Studies that explicitly analyze students' proof ability based on each APOS conception are still limited. While some studies have used APOS theory to examine students' cognitive processes in mathematical proof, most do not provide a detailed, systematic analysis within the framework (Mahfudhoh, Saniya, & Kohar, 2023; Syamsuri & Marethi, 2018; Ubah & Bansilal, 2018). This limitation makes it difficult to identify students' specific cognitive difficulties in constructing proofs in Real Analysis (Mukuka & Tatira, 2025; Weber, 2001). Therefore, this study aims to fill this gap by investigating students' proof ability in Real Analysis using the APOS framework.

Based on this background, students' ability to prove Real Analysis problems can be investigated using the APOS approach. This approach allows students to determine their level of understanding. Therefore, this study aims to investigate pre-service mathematics teachers' ability to construct proofs in Real Analysis based on APOS conceptions (Action, Process, Object, and Schema). The results of this study can be used as a reference in investigating the location of students' difficulties in proving Real Analysis problems, whether they are in the conception of action, process, object, or schema. This can also inform the development of learning models that improve students' ability to prove more effectively and provide material for assessing the quality of students' thinking processes in proving.

METHOD

This study uses a qualitative case study approach to explore pre-service mathematics teachers' ability to prove Real Analysis problems using the APOS (Action, Process, Object, Schema) framework. The case study design was chosen because it allows for an in-depth investigation of students' reasoning processes and cognitive structures in Real Analysis, providing a detailed understanding of how students construct proofs. The research subjects consisted of 59 students in the mathematics education study program at a university in Palembang, South Sumatera, Indonesia. Participants were purposively selected (Ahmad & Wilkins, 2024; Palinkas et al., 2015), namely students taking a Real Analysis course in their fifth semester and having completed prerequisite mathematics courses. This was to ensure that they had the basic knowledge necessary to solve proof-based problems.

The main instrument used was a written test that consisted of four proof questions. It had been designed to identify students' abilities at each phase of the APOS framework. Each item was scored using a rubric that evaluated the accuracy, logical coherence, and completeness of the solution (A. J. Stylianides, 2007; Weber, 2001). The test items were developed based on the Real Analysis syllabus and reviewed by three mathematics education experts to ensure content validity. Because the instrument

was self-developed and not derived from a standardized test, the researchers conducted validity and reliability testing before implementation. Experts evaluated each item using the Item-Content Validity Index (I-CVI) and the Scale-Content Validity Index (S-CVI) (Takom et al., 2025; Yusoff, 2019), resulting in I-CVI = 1 and S-CVI = 1, indicating that all items were fully valid. Construct validity was further examined using the Pearson Product-Moment correlation (DeVellis & Thorpe, 2021), where the r -statistics for the four items were 0.515, 0.727, 0.926, and 0.724, all exceeding the r -critical value ($n = 18$) of 0.468, which confirms the items' validity. Reliability of the test was assessed using Cronbach's Alpha, which resulted in $\alpha = 0.65$; although slightly below the conventional threshold of 0.7, this value is considered acceptable for exploratory studies with a small number of items (DeVellis & Thorpe, 2021; Zakariya, 2022). This test explores how students understand and construct proofs and reveals common errors and reasoning strategies.

The data from the four-problem test were then analyzed qualitatively using content and thematic analysis techniques (Braun & Clarke, 2006; Özden, 2024; Vaismoradi et al., 2013). This technique identified the thought patterns and APOS conceptions that emerged in students' proof-proving process. The analysis followed several steps: first, students' responses were read multiple times to gain familiarity and identify initial patterns; second, coding was conducted to label specific reasoning strategies, steps, and errors; third, codes were categorized according to the four APOS levels: Action, Process, Object, and Schema; and finally, the results were interpreted to identify recurring patterns and variations in students' proof abilities. The analysis focused on the methods and errors in proving. The analysis framework for categorizing students' responses is as follows: *Action conception*: Students apply the proof steps directly without in-depth reflection on why they apply them. *Process conception*: Students begin to understand the logical sequence of the proof steps and can visualize how the proof works as a whole. *Object conception*: Students consolidate the entire proof process into a single concept or "object," and are able to effectively use the proof method to prove similar or analogous problems. *Schema conception*: Students develop a comprehensive understanding of when and how to use different types of proofs, as well as connecting different mathematical concepts, or using the proof method as a strategic tool to prove various types of mathematical problems (Afgani, Suryadi, & Dahlan, 2017; Anwar & Abdillah, 2018; Arnon et al., 2014).

To ensure data validity, the researchers used internal triangulation by comparing participants' responses (Creswell & Poth, 2017; Morgan, 2024). In addition, colleagues peer-checked the analysis results to minimize subjective bias. The team discussed any discrepancies in coding and categorization until consensus was reached (Coleman et al., 2024; Miles et al., 2018). These procedures provide a more rigorous and trustworthy picture of students' proof-proving abilities based on the APOS framework.

RESULTS

This section reports the empirical findings derived from students' written responses. The discussion section interprets these findings and compares them with previous studies are presented in the discussion section. This study's results were obtained by analyzing the answers of pre-service mathematics teachers in semester 5 to 4 Real Analysis test questions. The first problem asks you to prove that the sum of a rational and an irrational number is irrational. Secondly about proving that $3 + 11 + \dots + (8n - 5) = 4n^2 - n$ for every $n \in \mathbb{N}$. The third is about showing $0 < c^2 < c < 1$, if $0 < c < 1$. The last problem is about showing $|x - y| < b - a$, if $a < x < b$ and $a < y < b$. To solve the first problem, students must realize that they must prove it by contradiction; the second by mathematical induction; and the third and fourth by direct proof. Next, for the APOS conceptual analysis framework, this study uses indicators, namely, first, students in action and process conception: whether they can solve proof problems using the method of contradiction, mathematical induction, or direct proof. Second, students in object conception of direct proof: whether they can solve two direct proof problems. Third, students in the scheme conception can solve all proof problems. This framework is appropriate because in the action conception, students apply algorithms or step-by-step procedures externally (following specific instructions or proof methods) without a deep understanding of the overall process or the flexibility to change their approach. In the process conception, students can carry out proof procedures mentally, understand the logical flow, and are flexible in choosing different proof methods. (Langi & Palayukan,

2024; Lapele et al., 2024). In the object conception, students can encapsulate the proof process into a self-contained cognitive object. Students can view the entire proof process (or the direct proof method in general) as an object that can be mentally manipulated or applied in more complex contexts. Solving several direct proof problems shows that students not only follow the steps of the proof but also treat the method as a conceptual object that can be used easily (Khairani, 2008; Windasari et al., 2020). In the schema conception, students can connect various proof methods (direct, contradiction, induction), understand when and why to use certain methods, and integrate these proof concepts with other mathematical concepts. The ability to solve all types of proof problems comprehensively strongly indicates the development of a mature schema (Buulele, Afriati, Desniarti, Sari, & Hutabarat, 2025; Khusna, 2020; Mulyono, 2011).

To make the categorization more explicit and observable in students' answers, we used the following operational indicators. We categorized students as having an action conception when they used specific examples, applied procedural steps without generalization, or presented an incomplete proof structure. We categorize students as having a process conception when they correctly identify and execute appropriate proof methods with mathematical reasoning. Students are considered to have reached the object conception when they are able to apply proof methods flexibly across similar problems. Finally, students are categorized as having a schema conception when they can integrate multiple proof methods across all given problems.

The research findings show that for the first problem, 5 of the 59 students gave irrelevant answers or did not answer, 40 of the 59 students only fulfilled the action conception, and the rest fulfilled the process conception. One student's answer that only fulfills the action conception shows action at the beginning of the proof by defining rational numbers by assuming a is a rational number and writing it as $a = p/q$, where p and q are integers and $q \neq 0$. For irrational numbers, they took one example of an irrational number, that is $\sqrt{2}$. Then, in the process of proving it, they immediately added up the numbers $1\frac{2}{3}$, which is a rational number with $\sqrt{2}$ (Figure 1). This shows that they are correct in defining rational numbers and stating that rational + irrational = irrational from a logical intuition, but their response is still in the form of an illustration or application example.

$a \in \mathbb{R}$ dapat dikatakan bilangan rasional ($a \in \mathbb{Q}$) jika bisa ditulis $a = \frac{p}{q}$ dimana $p, q \in \mathbb{Z}$ dan $q \neq 0$ misal : $1 \in \mathbb{Q} = \frac{1}{1}$ bilangan rasional karena $a \in \mathbb{R}$ salah satu contoh bilangan irrasional $\sqrt{2}$, Jadi jika, rasional + irrasional = irrasional $a = 1\frac{2}{3} + \sqrt{2}$ $a = \frac{5}{3} + \sqrt{2}$	Translate: $a \in \mathbb{R}$ can be said to be a rational number ($a \in \mathbb{Q}$) if it can be written as: $a = p/q$ where $p, q \in \mathbb{Z}$ and $q \neq 0$ example: $1 \in \mathbb{Q} = 1/1$ is a rational number because $a \in \mathbb{R}$ One example of an irrational number is $\sqrt{2}$ So, if Rational + irrational = irrational $a = 1\frac{2}{3} + \sqrt{2}$ $a = \frac{5}{3} + \sqrt{2}$
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Figure 1. Students' Answer in Action Conception on Problem No. 1

Based on the written response in Figure 1, the student used a specific example of a rational and an irrational number, then added them together, without proceeding to a general proof. This response is categorized as action conception that meets the operational indicator, namely the use of particular cases without generalization and the absence of a complete proof structure.

For the second problem, 15 of the 59 students gave irrelevant answers or did not answer, 31 of the 59 students only fulfilled the action conception, and 13 of the 59 students fulfilled the process conception. One of the students' answers that only fulfills the action conception shows the action at the

beginning of the proof by trying $n = 1$ on the left and right sides, then assuming it is true for $n = k$ so that $3 + 11 + \dots + (8k - 5) = 4k^2 - k$. When they proved to $n = k + 1$, they stated that each side added $n = k + 1$ and wrote $3 + 11 + \dots + (8k - 5) + (8(k + 1) - 5) = 4k^2 - k + 4(k + 1)^2 - (k + 1)$. This makes the two sides unequal. (Figure 2). The error lies in adding $n = k + 1$ to each side. They should be quite transformed $3 + 11 + \dots + (8n - 5) = 4n^2 - n$ by substituting $n = k + 1$ to become $3 + 11 + \dots + (8(k+1) - 5) = 4(k+1)^2 - (k+1)$. Then, they simply transformed the series of numbers on the left side, $3 + 11 + \dots + (8k - 5) = 4k^2 - k$, so that the left and right sides are equal. This shows that they have demonstrated basic algebraic manipulation skills, understand the transition process from $n=k$ to $n=k+1$, understand that the left and right sides must be manipulated in such a way as to achieve an equivalent form, but are constrained in proving that the result of simplifying the left side is exactly the same as the original formula form if n is replaced by $k+1$.

Untuk $n = k$, didapat

$$3 + 11 + \dots + (8k - 5) = 4k^2 - k$$

untuk $n = k+1$, masing-masing ruas ditambah $n = k+1$, didapat

$$3 + 11 + \dots + (8k - 5) + (8(k+1) - 5) = (4k^2 - k + (4(k+1)^2 - (k+1)))$$

$$4k^2 - k + 8k + 8 - 5 = 4k^2 - k + (4(k^2 + 2k + 1) - k - 1)$$

Translate:

For $n = k$, it is obtained

$$3 + 11 + \dots + (8k - 5) = 4k^2 - k$$

For $n = k+1$, each side is added by $n = k+1$, it is obtained

$$3 + 11 + \dots + (8k - 5) + (8(k+1) - 5) = (4k^2 - k + (4(k+1)^2 - (k+1)))$$

$$4k^2 - k + 8k + 8 - 5 = 4k^2 - k + (4(k^2 + 2k + 1) - k - 1)$$

Figure 2. Students' Answer in Action Conception on Problem No. 2

Based on the written response in Figure 2, the student has demonstrated the process of proving mathematical induction but made an error in the proof process for $n = k + 1$, so the proof is generally incomplete. This answer is classified as action conception, where students follow procedural steps but demonstrate incomplete or incorrect transformations in the proof process.

For the third problem, 19 of the 59 students gave irrelevant or no answers, 34 of the 59 students showed only action conception, and 6 of the 59 showed process conception. One student's answer that only fulfilled the action conception showed the action at the beginning of the proof by assuming $c = 0.2$ to get $c^2 = 0.04$, then they ordered the results obtained between 0 and 1 (Figure 3). This shows that their abilities are limited to performing arithmetic calculations (squares) and comparing decimal numbers directly; they have begun to internalize the rule that squaring a number between 0 and 1 will produce a smaller number, but cannot yet see that this process applies to all values of c in that range without doing explicit calculations.

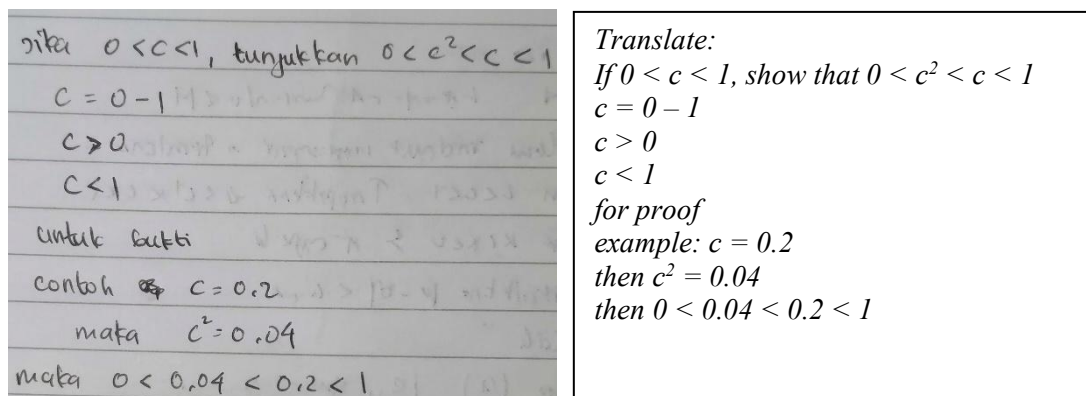
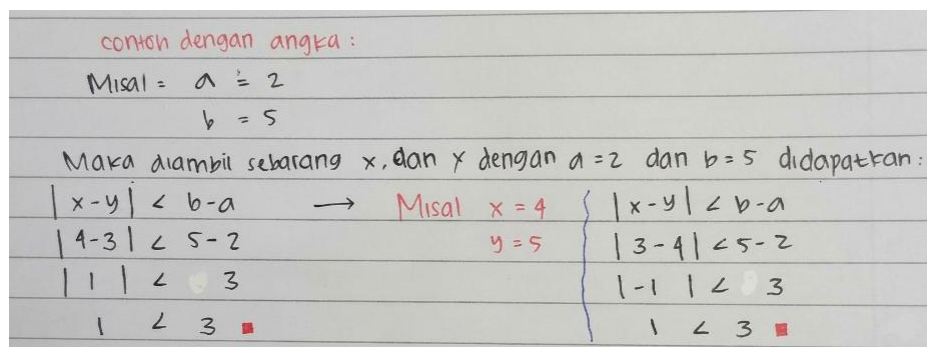


Figure 3. Students' Answer in Action Conception on Problem No. 3

Based on the written response in Figure 3, the student used a single example number and squared it, without continuing with a general proof. This response meets the criteria for action conception, where the student applies a single example and does not construct a general argument for all values.

For the fourth problem, 23 of the 59 students gave irrelevant or no answers, 36 of the 59 students only fulfilled the action conception, and none of the 59 fulfilled the process conception. One student's answer that only fulfilled the action conception showed the action at the beginning of the proof by assuming $a = 2$, $b = 5$, $x = 4$, and $y = 5$, then substituting them into the absolute inequality (Figure 4). This shows that their understanding relies on concrete number manipulation. They are beginning to understand that absolute value always makes the difference positive, and that the difference is always smaller than the interval width. However, they still need numerical examples to confirm this mental process. They are not yet able to see $|x-y|$ as the distance between two points and $(b-a)$ as the total length of the interval. Their understanding of inequality and absolute value is still fragmented. They know how to calculate them, but have not yet connected it to the concept of distance or geometry on the number line.



Translate:
 Example with numbers:
 Assume $a = 2$
 $b = 5$
 so, taking any arbitrary x and y with $a = 2$ and $b = 5$, we obtain
 $|x - y| < b - a$ Assume $x = 4$ $|x - y| < b - a$
 $|4 - 3| < 5 - 2$ $y = 5$ $|3 - 4| < 5 - 2$
 $|1| < 3$ $|-1| < 3$
 $1 < 3$ $1 < 3$

Figure 4. Students' Answer in Action Conception on Problem No. 4

Based on the written response in Figure 4, the student used a specific example of a natural number and substituted it for the variable in the absolute inequality, without proceeding to a general proof. This answer is categorized as action conception, where reasoning is limited to substituting specific values rather than forming a general inequality proof.

DISCUSSION

Based on the findings, most pre-service mathematics teachers' proof abilities on Real Analysis problems remain in the action conception; only a few have reached the process conception, and none have reached the object and schema conception. Several factors may influence this tendency. First, students' previous learning experiences may have emphasized procedural tasks over formal proof construction, leading them to use example-based arguments (Harel & Sowder, 1998; Weber, 2001). Second, limited understanding of proof methods may prevent students from knowing the appropriate strategy for unfamiliar problems (Mukuka & Tatira, 2025; A. J. Stylianides & Stylianides, 2022). Third, they may have difficulty recognizing abstract mathematical structures, which can hinder the transition from an action conception to a process conception (Arnon et al., 2014).

Findings from the analysis of students' responses to the first problem show that students proved the problem without deep reflection on its general validity. They also did not realize that the method of contradiction was required to prove the problem. When students used some examples to prove a statement, they were simply applying or observing the procedure to individual cases. They had not yet internalized the concept abstractly and did not see the logical connections that apply more generally. This indicates that students' understanding is still at a surface level, where mathematical truth is perceived as empirical verification rather than logical necessity. From an APOS perspective, this suggests that students have not yet interiorized actions into a coherent process. Students who use some examples do not demonstrate an understanding of the validity of mathematical proof as a whole. A valid mathematical proof must be general, not just true for a few cases. Students who rely only on examples usually cannot coordinate these actions into complete logical reasoning that applies to all possible cases (Harry & Cahya, 2019; Mahfudhoh et al., 2023; Maullina & Setyaningsih, 2023). On the other hand, students who fulfill the process conception realize that proving that the sum of rational and irrational numbers is irrational must use the contradiction method, because in the direct proof method, there is a problem in determining the general form of irrational numbers (Chamberlain & Vidakovic, 2021; Harel & Sowder, 1998; Weber, 2001). The emergence of process conception indicates that some students have begun to internalize proof procedures and understand their logical structure, although this understanding is not yet fully stable across different types of proofs (Mahfudhoh et al., 2023; Shinariko et al., 2022). Similarly, analysis of students' responses to the second problem shows that they recognized that mathematical induction must be used to prove the problem, but the final step to complete the proof process is problematic. This difficulty may stem from students' partial understanding of the inductive step, particularly in transforming expressions from $n = k$ to $n = k + 1$, which requires both algebraic manipulation and logical reasoning. Meanwhile, students who grasp the process conception not only recognize that, in prove the problem, they must use mathematical induction, but can also complete the proof for $n = k + 1$.

Furthermore, analysis of students' responses to the third problem shows that they proved it without deep reflection on its general validity. They also did not recognize that proving the problem would involve the theorem $a > c$ if $a > b$ and $b > c$, $a, b, c \in \mathbb{R}$, and the theorem $ca > cb$ if $a > b$ and $c > 0$, $a, b, c \in \mathbb{R}$. The inability to apply relevant theorems suggests that students may have fragmented knowledge of Real Analysis concepts, where definitions and theorems are not yet connected into a coherent structure (Mukuka & Tatira, 2025; Sevgi Morali & Filiz, 2023). In contrast, students who adopt the process conception recognize that, in prove the problem, they can use a direct method involving theorems about the order properties of real numbers. Similarly, in the fourth problem, students proved it without deep reflection on its general validity. They also failed to recognize that proving the problem requires the definition and theorem of absolute value, as well as the theorem on the order properties of real numbers. In contrast, students who develop process conception recognize that, in proving the

problem, they can use a direct method involving definitions and theorems (Mahfudhoh et al., 2023; Shinariko et al., 2022).

These findings were in line with the results of previous research, which reported that pre-service teachers experienced difficulties in understanding and applying formal proof methods (Mukuka & Tatira, 2025). Pre-service teachers also tend to make incorrect decisions about theorems and proofs and sometimes cannot distinguish between terms used in proofs. Students tend to learn proof through memorization rather than conceptual understanding, which implicitly indicates that their understanding may be low (action conception) and rarely reaches a deep level of understanding (object/schema) (Sevgi Morali & Filiz, 2023). In the research subjects of pre-service elementary school teachers, many of them experienced difficulties in developing a deep understanding (achieving object or schema conceptions) of fractions and tended to remain at action conception or incomplete process conception (Ubah & Bansilal, 2018). In addition, pre-service mathematics teachers lack adequate mental structures and do not reach the expected learning levels in the context of APOS theory. This indicates that many of them remain at the lower conception (action or process) and rarely reach the object or schema conception (Afgani et al., 2017; Urhan & Dost, 2018).

To achieve object conception, students must be able to prove it using a method in similar problems. This is facilitated in questions 3 and 4. Both questions can be proven using the direct method. However, the findings showed that no student could prove questions 3 and 4 simultaneously. Meanwhile, to fulfill the schema conception, students must be able to prove all four questions using various proof methods. Findings from the students' answers indicate that no student could prove all the questions. This indicates that they were not yet able to use all types of proof schema on Real Analysis problems. One finding showed that students who could solve proof problems using mathematical induction still made errors or struggled with other methods, such as direct or proof by contradiction. Cases like this fall into the process conception category, or may still be in transition towards the object conception, but have not yet reached the mature schema conception (Palla et al., 2012; Tatira, 2023; Tiros, 1999; Weber, 2001).

Successfully using mathematical induction indicates that students have internalized the process into a coordinated sequence. They can perform inductive reasoning mentally and understand its complete logical flow. Mastery of a complex proof method, such as induction, is often associated with the process conception for that proof concept. Weakness in using other methods (direct or contradiction) indicates that students have not fully encapsulated the general proof method into a flexible cognitive object. They may have mastered induction as an isolated process, but are not yet able to see the entire method of proof as an object that can be manipulated, compared, or applied flexibly in various contexts. (Mahfudhoh et al., 2023; Shinariko et al., 2022). If they are already at the Object conception, they should have a more solid understanding of the various methods and be flexible in choosing and using them correctly. Meanwhile, to achieve a schema conception, students must be able to engage with an organized network of all actions, processes, and objects related to mathematical proof as a whole. Students with a mature proof schema will understand the relationship between induction, direct proof, and contradiction, and know when and how to apply each method effectively and correctly. Inadequacies in other methods clearly indicate that their proof schemas are incomplete or poorly integrated. (Anwar & Abdillah, 2018; Aron et al., 2014; Harel & Sowder, 1998; Sefik et al., 2021; Suwanto et al., 2017).

These findings have important implications for the teaching of proof in Real Analysis courses. Instruction should not only focus on procedural mastery of proof techniques, but also emphasize conceptual understanding and the connections between different proof methods (Hanna, 2020; Rodríguez-Nieto et al., 2023; A. J. Stylianides & Stylianides, 2022). Learning activities that encourage students to compare, reflect on, and justify different proof strategies may help facilitate the transition from action to process and eventually to object and schema conceptions (Anwar & Abdillah, 2018; Aron et al., 2014; Harel & Sowder, 1998). Furthermore, this study contributes to the literature by providing empirical evidence that proof ability develops unevenly across proof methods within the APOS framework (Mukuka & Tatira, 2025). This suggests that mastery of one method (e.g., induction) does not automatically imply a well-developed proof schema, highlighting the need for more integrated instructional approaches (Shinariko et al., 2022; A. J. Stylianides & Stylianides, 2022).

CONCLUSION

Based on the results of the data analysis and discussion that have been carried out, this study concludes that the ability of pre-service mathematics teachers to prove Real Analysis problems using the APOS conception approach was that most of them are still in the action conception, only a few have reached the process conception, and none have fulfilled the object and schema conception. The key finding shows that they proved the problem without deep reflection. They did not recognize which method was required to prove it. They used some examples to prove a statement and simply applied or observed the procedure to individual cases. They had not yet internalized the concept abstractly or encapsulated the proof process as a conceptual object, and they did not see the logical connections that apply more generally. They also did not realize that proving the problem would involve definitions or theorems, not just completing the proof process. The results suggest that pre-service mathematics teachers still have a weak grasp of mathematical logic and algebraic manipulation, and they lack understanding of how to use definitions, axioms, and theorems in the proof process. For this reason, the curriculum needs to strengthen prerequisite abilities before studying Real Analysis and adopt a learning model that improves mathematical proof abilities not only in Real Analysis but in other mathematical areas as well. However, this study has limitations because it involved only 59 pre-service mathematics teachers from one study program at a university and analyzed and described only students' answers to four Real Analysis proof problems. Therefore, future research should include proof problems that require various proof methods, conduct tests more than once, and develop additional research instruments.

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