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Representasi Visual Mahasiswa S1 Pendidikan Matematika tentang Fungsi Multivariabel

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Abstrak

Representasi konsep matematika sangat penting dalam proses belajar mengajar matematika. Representasi visual digunakan untuk menyajikan konsep matematika untuk memecahkan, mengeksplorasi atau menjelaskan konsep. Tujuan dari penelitian ini adalah untuk menguji representasi visual mahasiswa S1 pendidikan matematika tentang fungsi multivariabel menggunakan tugas menjodohkan item. Penelitian ini merupakan penelitian kualitatif yang dilakukan terhadap 10 mahasiswa pendidikan matematika sarjana tahun kedua di sebuah Universitas di Zimbabwe. Siswa mencocokkan fungsi beberapa variabel yang direpresentasikan dalam bentuk simbolik dan grafik. Hasil penelitian menunjukkan bahwa sebagian besar siswa mampu mencocokkan fungsi dengan representasi grafisnya. Namun, sebagian besar siswa mengalami kesulitan dalam mengkomunikasikan informasi matematika dan kesulitan ini berasal dari masalah bahasa dan pemahaman konseptual.

Kata Kunci: konsep kalkulus multivariabel, latihan mencocokkan item, pemikiran visual, representasi grafis, representasi semiotic, representasi visual.

Undergraduate Mathematics Education Students' Visual Representations of Multivariable Functions

Abstract

Representations of mathematical concepts are very important in the teaching and learning of mathematics. Visual representations are used to present mathematical concepts to solve, explore or explain the concepts. The purpose of this study was to examine undergraduate mathematics education students' visual representations of multivariable functions using matching items tasks. The study was qualitative research conducted with 10 second-year undergraduate mathematics education students at a University in Zimbabwe. The students matched the functions of several variables which are represented in symbolic and graphic forms. The result showed that most of the students were able to match the functions with their graphical representations. However, most of the students had difficulties in communicating mathematical information and these difficulties emanate from language problems and conceptual understanding.

Keywords*: graphical representations; matching items exercise; multivariable calculus concepts; semiotic representations; visual representations; visual thinking*

INTRODUCTION

Representation of mathematical concepts is very important in the teaching and learning of mathematics. The ability to represent a mathematical situation in different forms is a very powerful tool in mathematics (Garzon & Casinillo, 2021; Golding, Hyde & Clark-Wilson, 2019; Kenan, 2018). Visual representations (for example, charts, graphs, pictures, diagrams, and numerical lines) can be used to solve, explore, or explain a mathematical situation. A person's visual representation of a mathematical situation is the person's external expression of the person's internal representation of the situation. Students can use visual representation to communicate their thoughts (Sanwidi, 2018).

Visual representation helps students to understand mathematical concepts (Novitasari, Usodo & Fitriana, 2021; Saifiyah & Retnawati, 2019), internalise the concepts, and establish connections between the concepts leading to conceptual understanding and creativity (Riccomini, Witzel & Deshpande, 2022; Kędra & Žakevičiūtė, 2019; Rumanová & Drábeková, 2019). In line with this thought, Duval (1995) noted that "there's no knowledge that can be mobilised by an individual without a representation activity" (p. 15).

The current trend in mathematics education puts more emphasis on conceptual understanding than on memorization of mathematical concepts. Visual representations are a key to conceptual understanding in mathematics (Davis, 2015; Murphy, 2019; Yilmaz & Argun, 2018). Snow (1967) quoting G.H. Hardy (1877-1947), an English mathematician, known for his work in number theory and mathematical analysis, said: "The mathematician's patterns, like the painter's or the poet's, must be beautiful; the ideas, like the colors or the words, must fit together in a harmonious way. Beauty is the first test: there is no permanent place in this world for ugly mathematics" (p.84). Contrary to this amazing statement many mathematics students perceive mathematics as a very dull, cold, and ugly subject. Cognitivism and constructivism relate learning to the involvement of the internal (psychological) and the external (physical) domains, hence the processing of information is a way of interaction between these two domains (Mnguni, 2014). The process of imagery involves the formation of mental images, figures, or likenesses of things. When a person thinks about a particular part of the external world, an image is evoked in the mind. The nature of the image differs from person to person (Tall, 1991). Visual representation is one process by which mental representations (imagery) can be seen. Mathematical objects can be represented through semiotic representations such as graphs, diagrams, symbols, and words (Duval, 1999). The mathematical activities involve the use of a variety of semiotic representation systems which comprise a natural language, the registers of numeric, algebraic, and symbolic notations, geometrical figures, and Cartesian graphs.

Mathematical functions can be represented visually using graphs and level curves (contour lines). A level curve of a function f(x,y) is the curve of points (x,y) where f(x,y) has constant values. Hence, a level curve of a function f(x,y) for the value c in the range of f(x,y) is the set of points such that f(x,y) = c (Strang & Herman, 2016).

Mathematicians have always used their "mind's eye" to visualize the abstract objects and processes that arise in all branches of mathematical research. Recently, mathematics educators have directed much attention to the issue of technology in the classroom and the curriculum, and to how technology could aid visualisation, particularly of graphs. This suggests that a description of mathematical visualisation should include not only mental imagery but also pictorial imagery in more concrete form by pencil and paper, calculator, and computer. Zimmerman and Cunningham (1991) provide an even broader and all-encompassing description of mathematical visualisation as inclined towards the ability to produce and understand how to use such a visual image correctly:

From the perspective of mathematical visualisation, the constraint that images must be manipulated mentally, without the aid of pencil and paper, seems artificial. In fact, in mathematical visualisation (visual representation) "what we are interested in is the student's ability to draw an appropriate diagram (with pencil and paper, or in some cases, with a computer) to represent a mathematical concept or problem and use the diagram to achieve understanding, and as an aid in problem-solving" (Zimmerman & Cunningham,1991, p.4).

Visual representation in mathematics is often associated with drawing pictures or diagrams as an aid in getting started on solving problems. However, visual representation has a much wider role to play in problem-solving, including supporting the development of ideas and facilitating communication of results and understanding. Some problems emphasise the use of visual representation to help students understand and develop a plan to solve a problem. In producing such a visual representation, the problem solver is identifying the key components of the problem and the relationships between them.

Some research studies have noted that some students and teachers have difficulties in the visual representations of mathematical concepts (Kashefi, Ismail & Yusof, 2010; Shongwe, 2021). In Zimbabwe, there has been no study that explored students' representations of mathematical concepts or functions in mathematics education research literature. Multivariable functions is one topic we observed that many students find difficult in the Zimbabwean university where the study was conducted. Hence, this study examined second-year undergraduate mathematics education students' visual translations of multivariable functions using matching items tasks.

METHOD

This qualitative case study was conducted with second-year mathematics education students at a University in Zimbabwe. Ten (6 males and 4 females) second-year mathematics students who had completed the course of Analysis participated in the study. The students were conveniently sampled from a group of fifty students. The researchers designed a matching exercise instrument with four tasks (see the appendix). The task required the students to match the functions with their graphs and level curves. The exercise tested how the students' switched the representations. That is, it examined the students' abilities to translate between the algebraic (symbolic) representation to the graphical and level curve representations and give reasons for their translations. The graphs and level curves were drawn using mathematical software. The tasks were used to explore three competencies namely, students' abilities to (a) match correctly the graph with its level curves, (b) give reasons for the matching, and (c) link geometrical concepts with analytic concepts.

RESULTS AND DISCUSSION

The data were analysed qualitatively. The analyses involved examining if the students correctly matched the functions with their graphical and level curve representations, and identifying the errors, if any, based on the reasons given by the students to support their matching of the functions with the graph and level curve representations. Radatz's (1979) error types framework was used to analyse the students' errors in explaining the reason for the matching. The errors were classified as one of or a combination of (1) language difficulties, (2) difficulties in processing iconic and visual representations of mathematical knowledge, (3) conceptual problems/Incomplete explanations, (4) Incorrect associations, and (5) application of irrelevant methods. The following is a summary of the results for each task.

1.1. Task A: Matching the function $z = x + 2y$ with its visual representations.

The visual representations of $z = x + 2y$ expected of the students are shown in Figure 1.

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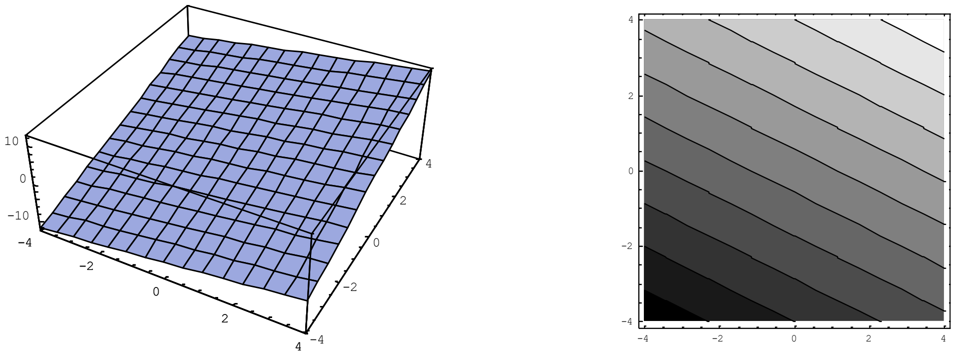


Graphical representation in space   Level curves

Figure 1. Visual representations of z = x + 2y

The result of the students on the task is presented in Table1.

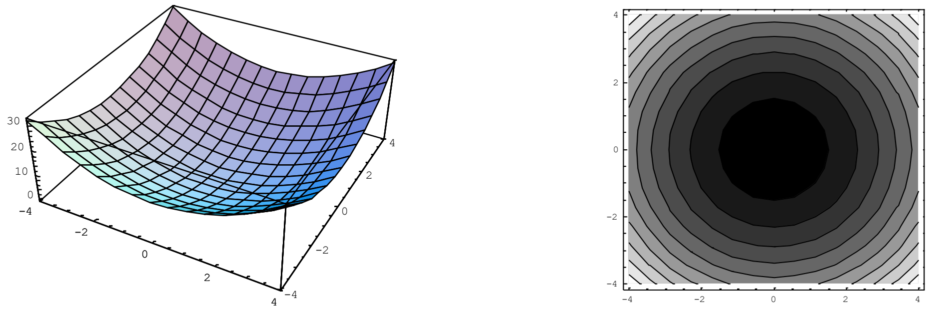
Table 1. Result on matching z = x + 2y with its graph and the level curve.

Student	Matching	Reason	Error type 1	2	3	4	5	Correct reason
A	Correct	It is an equation of a plane			✓			
B	Correct	The level curves are lines			✓			
C	Correct	Lines are linear curves			✓			
D	Correct	x+2y is represented on the z-axis on the graph			✓			
E	Correct	The level curves look like linear functions of the form x+2y = c.						✓
F	Correct	Because it is a linear function			✓			
G	Correct	f(x,y) is a linear function			✓			
H	Incorrect	-		✓				
I	Correct	It is a plane with parallel lines			✓			
J	Incorrect	Plane		✓	✓			

Eight out of ten students correctly matched the function with its graph and level curves. However, only one of the eight students was able to justify the matching. Two students had difficulties in processing visual representations of mathematical knowledge. Eight students had conceptual difficulties and their responses were incomplete.

1.2. Task B: Matching the function $z = x^2 + y^2$ with its visual representations

The visual representations of $z = x^2 + y^2$ expected of the students are shown in Figure 2.



Graphical representation in space | Level curves

Figure 2. Visual representations of $z = x^2 + y^2$

The result of the students on the task is presented in Table 2.

Table 2. The students' results on matching $z = x^2 + y^2$ with its graph and the level curve.

Student	Matching	Reason	Error type 1	2	3	4	5	Correct reason
A	Correct	It is a parabola centered at (0,0,0)			✓			
B	Correct	Circle, radius 2, center (0,0)			✓			
C	Correct	The graph drawn represents z = x2 + y2			✓			
D	Correct	Parabola			✓			
E	Correct	The level curves x2 +y2 = c are concentric circles.						✓
F	Correct	It is the equation of a circle.			✓			
G	Incorrect	-		✓				
H	Correct	f(x,y) is paraboloid			✓			
I	Correct	f(x,y) is a three-dimensional shape like a paraboloid			✓			
J	Correct	f(x,y) suits the equation of a circle and the function is a parabola.			✓			

Nine students out of the ten correctly matched the function with its graphical representations. Only one student produced a sound justification for why the match was done. Eight students produced incomplete reasons for matching and some of the reasons lacked conceptual understanding. One student had difficulties in processing visual representations of mathematical knowledge.

1.3. Task C: Matching the function $z = \frac{1}{x^2+y^2+1}$ *with its visual representations.*

The visual representations of the function are shown in Figure 3.

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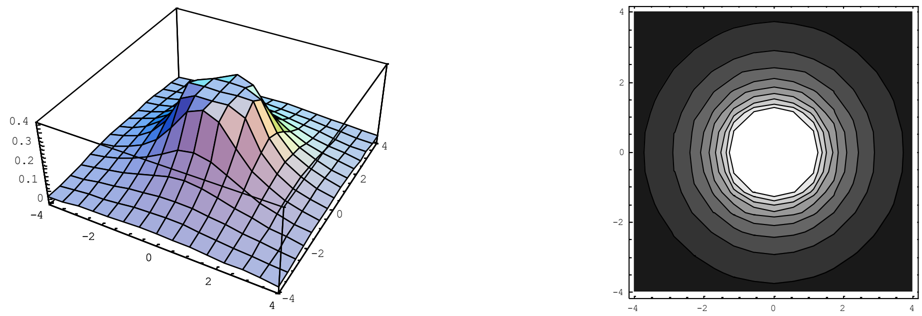


Graphical representation in space — Level curves

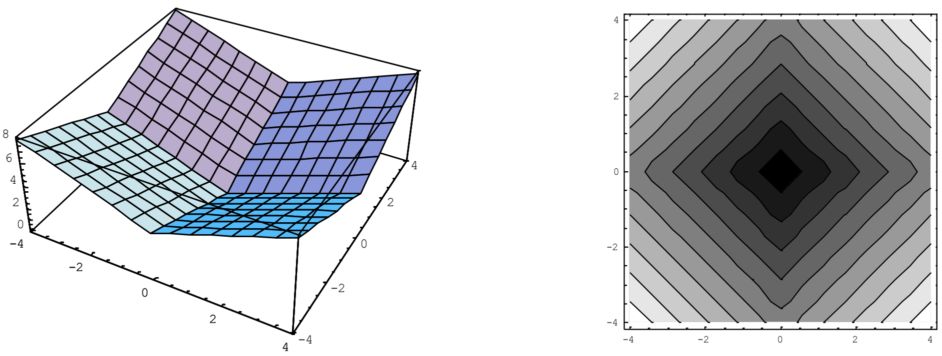
Figure 3. Visual representations of $z = \frac{1}{x^2+y^2+1}$

The students' performances on task A are presented in Table 3. Seven students were able to produce correct matching. Three students showed difficulties in processing visual representations of mathematical knowledge. Nine students produced incomplete statements to justify the reasons for matching. Only one student produced a sound justification for the correct matching of the function with its graphical representations.

Table 3. The students' results on the matching of $z = \frac{1}{x^2+y^2+1}$ with its graph and level curve.

Student	Matching	Reason	Error type 1	Error type 2	Error type 3	Error type 4	Error type 5	Correct reason
A	Correct	It is an inverse function			✓	✓		
B	Correct	It is a graph of x^2+y^2+1 which is transformed in the z-axis			✓			
C	Correct	Inverse function			✓	✓		
D	Correct	*f(x,y)* shows an inverse of three dimensional circular like object which can be conelike.	✓		✓	✓		
E	Correct	The graph looks like a hilltop with a maximum value at (0,0,1) and the level curves are concentric circles ($0< c < 1$)						✓
F	Correct	It is the inverse of a parabola			✓	✓		
G	Incorrect	It is a circle		✓	✓			
H	Incorrect	*f(x,y)* is the inverse of a parabola		✓	✓			
I	Correct	The inverse of a conical function	✓		✓	✓		
J	Incorrect	The function is dome shaped.		✓				

1.4. Task D: Matching the function $z = |x| + |y|$ with its visual representations.
The expected visual representations of the function are shown in Figure 4.



Graphical representation in space   Level curves

Figure 4. The graph and level curve of the function $z = |x| + |y|$

The students' performances on the task are presented in Table 4.

Table 4. The students' results on matching z = |x| +|y| with its graph and level curve.

Student	Matching	Reason	Error type 1	2	3	4	5	Correct reason
A	Correct	The function z collects all points on the x and y axes both negative and positive			✓			
B	Correct	z has both positive and negative			✓			
C	Correct	Modulus function			✓			
D	Correct	Since f(x,y) is a modulus function			✓			
E	Correct	The level curves |x|+|y| = c take the form of the form of the linear function of the form x+y = c in the positive x-axis.						✓
F	Correct	The graph shows the same lines which are similar to a graph of the function f(x) = |x|, therefore on contour graphics, it becomes very clear that it deals with modulus.	✓					✓
G	Correct	f(x) =|x|			✓			
H	Correct	It is a modulus function			✓			
I	Correct	f(x,y) is a function in 3 dimension			✓			
J	Correct	It looks like folded paper.			✓			

All students correctly matched the function with its graphical representations. Eight students gave incomplete justifications for the matching. Two were able to give sound arguments to justify their matching. One of those two had language difficulties in expressing the reason for matching.

The results of the students' performances on the visual representations matching tasks are summarised in Figure 5. The result showed that most of the students were able to process visual representation of mathematical functions and consequently it can be inferred from the results that visual representation of mathematical knowledge can be processed easily. Matching a function with graphical representations promotes visual thinking. Students in this research were able to visually link the function in three variables with its two graphical representations. This positive observation was also noted by Sword (2005) who posited that visual-spatial thinking is complex, rich, textured, detailed, and imaginative. With visual thinking, the information is processed instantly, just by looking at the picture.

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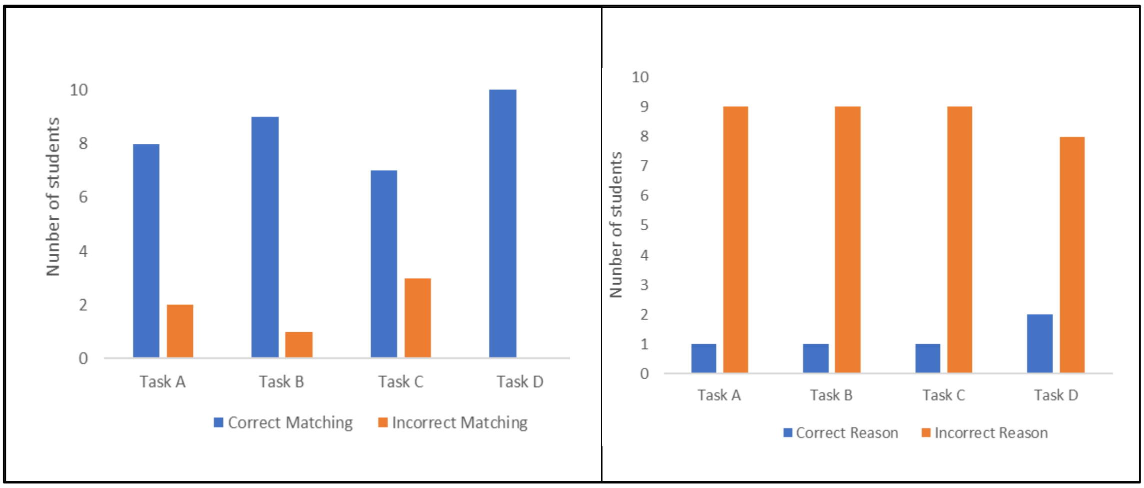

Figure 5. The students' performances on visual representations of the multivariable functions' tasks.

Furthermore, the result showed that most students had difficulties communicating mathematical information. The matching exercise was easy but describing the matching with cogent reasons was a difficult issue for the students. It was observed that some students had language problems in communicating mathematical ideas, for example on task C the reason given by student D was as follows: "f(x,y) shows an inverse of three dimensional circular like object which can be a conelike". Here mathematical language was not properly communicated because three-dimensional objects have no inverse, but algebraic representations of functions can admit inverse functions. The student confused the concept of reciprocal functions with inverse functions. On the same note, it was also notable that some students used metaphors when visualizing some graphs of functions. Terms like 'conelike'. 'hilltop' and 'folded paper' were used to describe the nature of some graphs of functions. Mathematics should be considered as a language that must be meaningful if students are to communicate mathematically and apply mathematics productively. Communication plays an important role in making mathematics meaningful; it enables students to construct links between their informal, intuitive notions and the abstract language and symbolism of mathematics. The idea of communicating mathematics at the undergraduate level was emphasized by Engelbrecht (2010) who listed the following stages in the growth of mathematical knowledge namely: (1) verbal, hand-waiving, informal explanation in English (or whatever language they speak) (2) Visually (drawing pictures), (3) Formal mathematical symbolism and (4) proving that a concept is valid, or that the result is true.

The mathematical activity involves the use of a variety of semiotic representation systems which comprise a natural language, the registers of numeric, algebraic, and symbolic notations, geometrical figures, and Cartesian graphs. It was observed from the responses that students had difficulties in making verbal descriptions to justify the reasons for matching the symbolic representations with graphical representations of functions. This agrees with the observation of Kashefi, Ismail and Yusof (2010) that students have difficulties in representing some three-dimensional shapes.

In this research, there were cases where some of the students communicated mathematical information excellently. For instance, the reasons for the matchings given by student E were correct. This student's reasoning went beyond the concept image to the concept definition. The ability to know the definition of the concept of level curves was helpful in the matching exercise. This student was able to link geometrical concepts with analytic concepts. Most students were not able to make this connection. However, some succeeded by associating the characteristics of single-variable functions. For example, the visual characteristics of the modulus function of a single variable and the visual characteristics of the graph of $f(x,y) = |x| + |y|$.

CONCLUSION

The result of this study showed that while most of the students were able to match the functions with their graphical representations, most of them could not give valid reasons for the matching. Most of them could not communicate mathematical information. The difficulties emanate from language problems and a lack of conceptual understanding of mathematical concepts. It is recommended that translating between the semiotic systems used to access mathematical knowledge be encouraged in mathematics teaching and learning. It is also recommended that mathematics education students should be encouraged to describe their concept images, in writing or verbally, to enhance their comprehension of mathematical concepts.

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